Diploma Programme
Programme du diplôme
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Mathematics: applications and interpretation
Higher level
Paper 3

Thursday 12 May 2022 (morning)

1 hour

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: applications and interpretation formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[55 marks]**.

5 pages

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Answer **both** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 28]

This question is about modelling the spread of a computer virus to predict the number of computers in a city which will be infected by the virus.

A systems analyst defines the following variables in a model:

- t is the number of days since the first computer was infected by the virus.
- $Q(t)$ is the total number of computers that have been infected up to and including day t.

The following data were collected:

t	10	15	20	25	30	35	40
$Q(t)$	20	90	403	1806	8070	32667	120146

(a) (i) Find the equation of the regression line of $Q(t)$ on t. [2]

(ii) Write down the value of r, Pearson's product-moment correlation coefficient. [1]

(iii) Explain why it would not be appropriate to conduct a hypothesis test on the value of r found in (a)(ii). [1]

A model for the early stage of the spread of the computer virus suggests that

$$Q'(t) = \beta N Q(t)$$

where N is the total number of computers in a city and β is a measure of how easily the virus is spreading between computers. Both N and β are assumed to be constant.

(b) (i) Find the general solution of the differential equation $Q'(t) = \beta N Q(t)$. [4]

(ii) Using the data in the table write down the equation for an appropriate non-linear regression model. [2]

(iii) Write down the value of R^2 for this model. [1]

(iv) Hence comment on the suitability of the model from (b)(ii) in comparison with the linear model found in part (a). [2]

(v) By considering large values of t write down one criticism of the model found in (b)(ii). [1]

(This question continues on the following page)

(Question 1 continued)

(c) Use your answer from part (b)(ii) to estimate the time taken for the number of infected computers to double. [2]

The data above are taken from city X which is estimated to have 2.6 million computers. The analyst looks at data for another city, Y. These data indicate a value of $\beta = 9.64 \times 10^{-8}$.

(d) Find in which city, X or Y, the computer virus is spreading more easily. Justify your answer using your results from part (b). [3]

An estimate for $Q'(t)$, $t \geq 5$, can be found by using the formula:

$$Q'(t) \approx \frac{Q(t+5) - Q(t-5)}{10}.$$

The following table shows estimates of $Q'(t)$ for city X at different values of t.

t	10	15	20	25	30	35	40
$Q(t)$	20	90	403	1806	8070	32 667	120 146
$Q'(t)$		a	171.6	766.7	b	11 207.6	

(e) Determine the value of a and of b. Give your answers correct to one decimal place. [2]

An improved model for $Q(t)$, which is valid for large values of t, is the logistic differential equation

$$Q'(t) = kQ(t)\left(1 - \frac{Q(t)}{L}\right)$$

where k and L are constants.

Based on this differential equation, the graph of $\frac{Q'(t)}{Q(t)}$ against $Q(t)$ is predicted to be a straight line.

(f) (i) Use linear regression to estimate the value of k and of L. [5]

(ii) The solution to the differential equation is given by

$$Q(t) = \frac{L}{1 + C\mathrm{e}^{-kt}}$$

where C is a constant.

Using your answer to part (f)(i), estimate the percentage of computers in city X that are expected to have been infected by the virus over a long period of time. [2]

2. [Maximum mark: 27]

This question is about a metropolitan area council planning a new town and the location of a new toxic waste dump.

A metropolitan area in a country is modelled as a square. The area has four towns, located at the corners of the square. All units are in kilometres with the x-coordinate representing the distance east and the y-coordinate representing the distance north from the origin at $(0, 0)$.

- Edison is modelled as being positioned at $\mathrm{E}(0, 40)$.
- Fermitown is modelled as being positioned at $\mathrm{F}(40, 40)$.
- Gaussville is modelled as being positioned at $\mathrm{G}(40, 0)$.
- Hamilton is modelled as being positioned at $\mathrm{H}(0, 0)$.

(a) The model assumes that each town is positioned at a single point. Describe possible circumstances in which this modelling assumption is reasonable. [1]

(b) Sketch a Voronoi diagram showing the regions within the metropolitan area that are closest to each town. [1]

The metropolitan area council decides to build a new town called Isaacopolis located at $\mathrm{I}(30, 20)$.

A new Voronoi diagram is to be created to include Isaacopolis. The equation of the perpendicular bisector of $[\mathrm{IE}]$ is $y = \frac{3}{2}x + \frac{15}{2}$.

(c) (i) Find the equation of the perpendicular bisector of $[\mathrm{IF}]$. [4]

(ii) Given that the coordinates of one vertex of the new Voronoi diagram are $(20, 37.5)$, find the coordinates of the other two vertices within the metropolitan area. [4]

(iii) Sketch this new Voronoi diagram showing the regions within the metropolitan area which are closest to each town. [2]

The metropolitan area is divided into districts based on the Voronoi regions found in part (c).

(d) A car departs from a point due north of Hamilton. It travels due east at constant speed to a destination point due North of Gaussville. It passes through the Edison, Isaacopolis and Fermitown districts. The car spends 30% of the travel time in the Isaacopolis district.

Find the distance between Gaussville and the car's destination point. [4]

(This question continues on the following page)

(Question 2 continued)

A toxic waste dump needs to be located within the metropolitan area. The council wants to locate it as far as possible from the nearest town.

(e) (i) Find the location of the toxic waste dump, given that this location is not on the edge of the metropolitan area. [4]

(ii) Make one possible criticism of the council's choice of location. [1]

(f) The toxic waste dump, T, is connected to the towns via a system of sewers.

The connections are represented in the following matrix, $\boldsymbol{M}$, where the order of rows and columns is $(\mathrm{E}, \mathrm{F}, \mathrm{G}, \mathrm{H}, \mathrm{I}, \mathrm{T})$.

$$\boldsymbol{M} = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}$$

A leak occurs from the toxic waste dump and travels through the sewers. The pollution takes one day to travel between locations that are directly connected.

The digit 1 in $\boldsymbol{M}$ represents a direct connection. The values of 1 in the leading diagonal of $\boldsymbol{M}$ mean that once a location is polluted it will stay polluted.

(i) Find which town is last to be polluted. Justify your answer. [3]

(ii) Write down the number of days it takes for the pollution to reach the last town. [1]

(iii) A sewer inspector needs to plan the shortest possible route through each of the connections between different locations. Determine an appropriate start point and an appropriate end point of the inspection route.

Note that the fact that each location is connected to itself does not correspond to a sewer that needs to be inspected. [2]

References: